Singular Liouville Equations on S^2: Sharp Inequalities and Existence Results

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Abstract

We prove a sharp Onofri-type inequality and non-existence of extremals for a Moser-Tudinger functional on S^2 in the presence of potential having positive order singularities. We also investigate the existence of critical points and give some sufficient conditions under symmetry or nondegeneracy assumptions.

1 Introduction

In this work we study sharp Onofri-type inequalities on the standard Euclidean sphere (S^2, g_0), and existence of critical points for a singular Moser-Trudinger functional. Given a smooth, closed surface Σ, and m points $p_1, \ldots, p_m \in \Sigma$, we consider the functional

$$J^h_\rho(u) = \frac{1}{2}\int_\Sigma |\nabla u|^2 dv_g + \frac{\rho}{|\Sigma|}\int_\Sigma u\, dv_g - \rho \log\left(\frac{1}{|\Sigma|}\int_\Sigma h e^u dv_g\right) \tag{1}$$

where h is a positive singular potential satisfying

$$h \in C^\infty(\Sigma\backslash\{p_1,\ldots,p_m\}) \qquad \text{and} \qquad h(x) \approx d(x,p_i)^{2\alpha} \text{ with } \alpha_i > -1 \text{ near } p_i, \tag{2}$$

$i = 1, \ldots, m$. Functionals of this this kind were first introduced, for the regular case $m = 0$, by Moser ([18], [19]), in connection to the study of the Gaussian curvature equation on compact surfaces and Nirenberg's problem on S^2. They also have a role in spectral analysis due to Polyakov's formula (see [23], [24], [22], [21]). In the case $m > 0$, the functional (1) appears in the problem of prescribing the Gaussian curvature of Riemannian metrics with conical singularities. We recall that a metric on Σ with conical singularities of order $\alpha_1, \ldots, \alpha_m > -1$ in $p_1, \ldots, p_m$, is a metric of the form $e^u g$ where g is smooth metric on Σ, and $u \in C^\infty(\Sigma\backslash\{p_1,\ldots,p_m\})$ satisfies

$$|u(x) + 2\alpha_i \log d(x,p_i)| \le C \qquad \text{near } p_i,\ i = 1, \ldots, m.$$

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It is possible to prove (see Proposition 2.1 in [3]) that a metric of this form has Gaussian curvature K if and only if u is a distributional solution of the Gaussian curvature equation

$$-\Delta_g u = 2Ke^u - 2K_g - 4\pi \sum_{i=1}^{m} \alpha_i \delta_{p_i}. \tag{3}$$

where K_g is the Gaussian curvature of (Σ, g). If $\chi(\Sigma) + \sum_{i=1}^{m} \alpha_i \neq 0$ and K_g is constant, (3) is equivalent to the singular Liouville equation

$$-\Delta_g u = \rho \left(\frac{Ke^u}{\int_\Sigma Ke^u dv_g} - \frac{1}{|\Sigma|} \right) - 4\pi \sum_{i=1}^{m} \alpha_i \left(\delta_{p_l} - \frac{1}{|\Sigma|} \right) \tag{4}$$

for

$$\rho = \rho_{geom} := 4\pi \left(\chi(\Sigma) + \sum_{i=1}^{m} \alpha_i \right). \tag{5}$$

Denoting by G the Green's function of (Σ, g), that is the solution of

$$\begin{cases} -\Delta_g G(x, \cdot) = \delta_x \quad \text{on } \Sigma \\ \int_\Sigma G(x,y) dv_g(y) = 0, \end{cases} \tag{6}$$

the change of variable $u \longleftrightarrow u + 4\pi \sum_{i=1}^{m} \alpha_i G(x, p_i)$ reduces (4) to

$$-\Delta_g u = \rho \left(\frac{he^u}{\int_\Sigma he^u dv_g} - \frac{1}{|\Sigma|} \right) \tag{7}$$

that is the Euler-Lagrange equation of the functional (1) corresponding to the potential

$$h(x) = Ke^{-4\pi \sum_{i=1}^{m} \alpha_i G_{p_i}}, \tag{8}$$

which satisfies (2). Equations (4) and (7) have also been widely studied in mathematical physics. For example, they appear in the description of Abelian vortices in Chern-Simmons-Higgs theory, and have applications in Superconductivity and Electroweak theory ([26], [14]). We refer to [4], [7], [8], [16], [6], [12], [13], for some recent existence results.

A fundamental role in the variational analysis of (1) is played by singular versions of the standard Moser-Trudinger inequality (see [18, 27]). In [27], Troyanov proved that, for every function h satisfying (2), there exists a constant $C = C(h, g, \Sigma)$ such that

$$\log \left(\frac{1}{|\Sigma|} \int_\Sigma he^{u-\overline{u}} dv_g \right) \leq \frac{1}{16\pi(1+\overline{\alpha})} \int_\Sigma |\nabla u|^2 dv_g + C(h, \Sigma, g) \tag{9}$$

$\forall\ u \in H^1(\Sigma)$, where $\overline{\alpha} = \min \left\{ 0, \min_{1 \leq i \leq m} \alpha_i \right\}$. In particular the functional J_ρ^h is bounded from below $\forall\ \rho \in (0, 8\pi(1+\overline{\alpha})]$ and is coercive for $\rho \in (0, 8\pi(1+\overline{\alpha}))$. Furthermore, it is possible to prove that the constant $\frac{1}{16\pi}$ is sharp, that is

$$\inf_{H^1(\Sigma)} J_\rho^h = -\infty \quad \forall\ \rho > 8\pi(1+\overline{\alpha}).$$

In the special case $m = 0$, $h \equiv 1$ and $(\Sigma, g) = (S^2, g_0)$, a sharp version of (9) was proved by Onofri in [20]:

Theorem A (Onofri's inequality [20]). *$\forall\, u \in H^1(S^2)$ we have*

$$\log\left(\frac{1}{4\pi}\int_{S^2} e^{u-\overline{u}} dv_{g_0}\right) \le \frac{1}{16\pi}\int_{S^2} |\nabla u|^2 dv_{g_0}$$

with equality holding if and only if $e^u g_0$ is a metric on S^2 with positive constant Gaussian curvature, or, equivalently, $u = \log|\det d\varphi| + c$ with $c \in \mathbb{R}$ and $\varphi : S^2 \longrightarrow S^2$ a conformal diffeomorphism of S^2.

Motivated by this result, in [17] we started the study Onofri-type inequalities and existence of energy-minimizing solutions on S^2 for the potential

$$h(x) = e^{-4\pi\sum_{i=1}^m \alpha_i G(x,p_i)}$$

(i.e. (8) with $K \equiv 1$), and we extended Theorem A to the cases $m = 1$, and $m = 2$ with $\min\{\alpha_1, \alpha_2\} < 0$.

Theorem B ([17])**.** *If $h = e^{-4\pi\alpha G_p}$ with $\alpha \neq 0$, then $\forall\, u \in H^1(\Sigma)$*

$$\log\left(\frac{1}{4\pi}\int_{S^2} he^{u-\overline{u}} dv_{g_0}\right) < \frac{1}{16\pi\min\{1, 1+\alpha\}}\int_{S^2} |\nabla u|^2 dv_{g_0} + \max\{\alpha, -\log(1+\alpha)\}.$$

Moreover equation (7) *has no solution for $\rho = 8\pi\min\{1, 1+\alpha\}$.*

Theorem C ([17])**.** *If $h = e^{-4\pi\alpha_1 G_p - 4\pi\alpha_2 G_{p_2}}$ with $p_2 = -p_1$, $\alpha_1 = \min\{\alpha_1, \alpha_2\} < 0$, then $\forall\, u \in H^1(\Sigma)$*

$$\log\left(\frac{1}{4\pi}\int_{S^2} he^{u-\overline{u}} dv_{g_0}\right) \le \frac{1}{16\pi(1+\alpha_1)}\int_{S^2} |\nabla u|^2 dv_{g_0} + \alpha_2 - \log(1+\alpha_1).$$

If $\alpha_1 \neq \alpha_2$ there is no function realizing equality and no solution of (7) *for $\rho = 8\pi(1+\alpha_1)$, while if $\alpha_1 = \alpha_2$ then equality holds for u if and only if the following equivalent conditions are satisfied:*

- *u is a solution of* (7) *for $\rho = 8\pi(1+\alpha_1)$.*
- *$he^u g$ is a metric with constant positive Gaussian curvature and conical singularities of order α_i in p_i, $i = 1, 2$.*
- *If π denotes the stereographic projection from p_1, then*

  $$u \circ \pi^{-1}(y) = 2\log\left(\frac{(1+|y|^2)^{1+\alpha_1}}{1+e^{\lambda}|y|^{2(1+\alpha_1)}}\right) + c$$

  for some λ, $c \in \mathbb{R}$.

We stress that the critical parameter $\rho = 8\pi(1+\overline{\alpha})$ is generally different from the geometric parameter (5) (except for some special cases, for example $m = 2$ and $\alpha_1 = \alpha_2 < 0$), thus critical points cannot always be interpreted in terms of metrics with prescribed curvature.

In this paper we will assume (8) with $\alpha_i \ge 0$ for $1 \le i \le m$ and

$$K \in C^\infty_+(S^2) := \left\{f \in C^\infty(S^2) \;:\; f(x) > 0 \quad \forall\, x \in S^2\right\}.$$

Our first result is a further extension of Onofri's inequality.

Theorem 1.1. *Assume that h satisfies* (8) *with $K \in C^\infty_+(S^2)$ and $\alpha_1, \ldots, \alpha_m \geq 0$, then*

$$\inf_{H^1(S^2)} J^h_{8\pi} = -8\pi \log \max_{S^2} h.$$

Moreover J^h_ρ has no minimum point, unless $\alpha_1 = \ldots = \alpha_m = 0$ (or, equivalently, $m = 0$) and K is constant.

Clearly, the sharp value of the constant $C(h, S^2, g_0)$ is given by

$$C(h, S^2, g_0) = -\frac{1}{8\pi(1+\overline{\alpha})} \inf_{H^1(S^2)} J^h_{8\pi(1+\overline{\alpha})},$$

thus Theorem 1.1 is equivalent to the following sharp inequality:

Corollary 1.1. *If h satisfies* (8) *with $K \in C^\infty_+(S^2)$ and $\alpha_1, \ldots, \alpha_m \geq 0$, then $\forall\, u \in H^1(S^2)$ we have*

$$\log\left(\frac{1}{4\pi}\int_{S^2} he^{u-\overline{u}} dv_{g_0}\right) \leq \frac{1}{16\pi}\int_{S^2} |\nabla u|^2 dv_{g_0} + \log \max_{S^2} h$$

with equality holding if and only if $m = 0$, K is constant and u realizes equality in Theorem A.

Theorem 1.1 states that $J^h_{8\pi}$ has no minimum point, but does not exclude the existence of different kinds on critical points. In contrast to Theorem C, if $\alpha_i > 0$ for $1 \leq i \leq m$, we will show that in many cases it is possible to find saddle points of $J^h_{8\pi}$. A simple example is given by the case in which h is axially symmetric. In this case an improved Moser-Trudinger inequality allows to minimize $J^h_{8\pi}$ in the class of axially symmetric functions and find a solution of (7).

Theorem 1.2. *Assume that h satisfies* (8) *with $m = 2$, $p_1 = -p_2$, $\min\{\alpha_1, \alpha_2\} = \alpha_1 > 0$ and $K \in C^\infty_+(S^2)$ axially symmetric with respect to the direction identified by p_1 and p_2. Then the Liouville equation* (7) *has an axially symmetric solution $\forall \rho \in (0, 8\pi(1+\alpha_1))$.*

In the last part of the paper we prove further general existence results using the Leray-Schauder degree theory introduced in [15], [10], [11], [12] and [13]. Solutions of (7) on the space

$$H_0 := \left\{u \in H^1(S^2) \ : \ \int_{S^2} u\, dv_{g_0} = 0\right\}. \tag{10}$$

can be obtained as solutions of $T_\rho(u) + u = 0$ where $T_\rho : H_0 \longrightarrow H_0$ is defined by

$$T_\rho(u) = \Delta^{-1}_{g_0}\left(\frac{he^u}{\int_{S^2} he^u dv_{g_0}} - \frac{1}{4\pi}\right). \tag{11}$$

In [13], Chen and Lin computed the Leray-Schauder degree

$$d_\rho = deg_{LS}(Id + T_\rho, 0, B_R(0)) \tag{12}$$

for

$$\rho \notin \Gamma(\alpha_1, \ldots, \alpha_n) := \left\{8\pi k_0 + 8\pi \sum_{i=1}^m k_i(1+\alpha_i) \ : \ k_0 \in \mathbb{N}, k_i \in \{0,1\}, \sum_{i=0}^m k_i > 0\right\}. \tag{13}$$

If $m \geq 2$, one has $d_\rho \neq 0$ for any $\rho \in (0, 8\pi(1+\overline{\alpha}))\backslash 8\pi\mathbb{N}$. While Theorem 1.1 implies blow-up as $\rho \nearrow 8\pi$, we can find solutions for $\rho = 8\pi$ by taking $\rho \searrow 8\pi$, provided the Laplacian of K is not too large at the critical points of h.

Theorem 1.3. *If h satisfies (8) with $K \in C_+^\infty(S^2)$, $m \geq 2$, $\alpha_1, \ldots, \alpha_m > 0$ and*

$$\Delta_{g_0} \log K(x) < \sum_{i=1}^{m} \alpha_i \tag{14}$$

$\forall\, x \in \Sigma$ such that $\nabla h(x) = 0$, then equation (7) has a solution for $\rho = 8\pi$.

The same strategy can be used for $\rho = 8k\pi$, with $k < 1 + \alpha_1$.

Theorem 1.4. *If h satisfies (8) with $K \in C_+^\infty(S^2)$, $m \geq 2$, $0 < \alpha_1 \leq \ldots \leq \alpha_m$ and*

$$\Delta_{g_0} \log K(x) < \sum_{i=1}^{m} \alpha_i + 2(1-k) \tag{15}$$

$\forall\, x \in S^2$, then equation (7) has a solution for $\rho = 8k\pi$, $k < 1 + \alpha_1$.

Note that Theorems 1.3 and 1.4 can be applied in the case $K \equiv 1$. If the sign condition (14) is not satisfied, then it is not possible to exclude blow-up of solutions as $\rho \longrightarrow 8\pi$. However, as it is pointed out in the introduction of [11], under some non-degeneracy assumptions on h, the Leray Schauder degree $d_{8\pi}$ is well defined and can be explicitly computed by taking into account the contributions of all the blowing-up families of solutions. In particular one can prove that $d_{8\pi} \neq 0$ under one of the following conditions.

Theorem 1.5. *Let h be a Morse function on $S^2 \backslash \{p_1, \ldots, p_m\}$ satisfying (8) with $K \in C_+^\infty(S^2)$, $m \geq 0$, $\alpha_1, \ldots, \alpha_m > 0$ and assume $\Delta_{g_0} \log h \neq 0$ at all the critical points of h. If h has r local maxima and s saddle points in which $\Delta_{g_0} h < 0$, then equation (7) has a solution for $\rho = 8\pi$ provided $r \neq s + 1$.*

Theorem 1.6. *Let h be a Morse function on $S^2 \backslash \{p_1, \ldots, p_m\}$ satisfying (8) with $K \in C_+^\infty(S^2)$, $m \geq 0$, $\alpha_1, \ldots, \alpha_m > 0$ and assume $\Delta_{g_0} \log h \neq 0$ at all the critical points of h. If h has r' local minima in $S^2 \backslash \{p_1, \ldots, p_m\}$ and s' saddle points in which $\Delta_{g_0} h > 0$, then equation (7) has a solution for $\rho = 8\pi$ provided $s' \neq r' + \overline{d}$, where*

$$\overline{d} := d_{8\pi+\varepsilon} = \begin{cases} 2 & m \geq 2, \\ 0 & m = 1, \\ -1 & m = 0. \end{cases}$$

In the regular case $m = 0$, Theorem 1.5 was first proved by Chang and Yang in [9] using a min-max scheme. A different proof was later given by Struwe [25] through a geometric flow approach.

2 Proof of the Main Results

The proof of Theorem 1.1 is a rather simple consequence of Theorem A.

Proof of Theorem 1.1. Let us consider

$$J^1_{8\pi}(u) := \frac{1}{2}\int_{S^2} |\nabla u|^2 dv_{g_0} + 2\int_{S^2} u\, dv_{g_0} - 8\pi \log\left(\frac{1}{4\pi}\int_{S^2} e^u dv_{g_0}\right).$$

By Theorem A we have $J^1_{8\pi}(u) \geq 0\ \forall\ u \in H^1(S^2)$. The condition $\alpha_1, \ldots, \alpha_m \geq 0$ guarantees $h \in C^0(S^2)$. Thus we have

$$J^h_{8\pi}(u) \geq \frac{1}{2}\int_{S^2} |\nabla u|^2 dv_{g_0} + 2\int_{S^2} u\, dv_{g_0} - 8\pi \log\left(\frac{1}{4\pi}\max_{S^2} h \int_{S^2} e^u dv_{g_0}\right) = \tag{16}$$

$$= J^1_{8\pi}(u) - 8\pi \log \max_{S^2} h \geq -8\pi \log \max_{S^2} h.$$

Since $e^u > 0$ on S^2, equality can hold only if

$$h \equiv \max_{S^2} h$$

which, by (8), is possible only if $\alpha_1 = \ldots = \alpha_m = 0$ and K is constant. To complete the proof it is sufficient to observe that the lower bound in (16) is sharp. Let us fix a point $p \in S^2$ such that $h(p) = \max_{S^2} h$, and consider the stereographic projection $\pi : S^2 \backslash \{p\} \longrightarrow \mathbb{R}^2$. For $t > 0$ we define $u_t := \log|\det d\varphi_t|$, where $\{\varphi_t\}_{t>0}$ is the family of conformal diffeomorphisms of S^2 that, in the local coordinates determined by π, corresponds to the family of dilations of $\mathbb{R}^2$, namely

$$\pi(\varphi_t(\pi^{-1}(y))) = ty \qquad \forall\, y \in \mathbb{R}^2.$$

By Theorem A, we have $J^1_{8\pi}(u_t) = 0\ \ \forall\, t > 0$. Moreover it is straightforward to verify that

$$\int_{S^2} e^{u_t} dv_{g_0} = 4\pi$$

and $e^{u_t} \rightharpoonup 4\pi\delta_p$ weakly as measures on S^2 for $t \to \infty$. Thus, one has

$$J^h_{8\pi}(u_t) = -8\pi \log\left(\frac{1}{4\pi}\int_{S^2} h e^{u_t} dv_{g_0}\right) \overset{t\to\infty}{\longrightarrow} -8\pi \log h(p) = -8\pi \log \max_{S^2} h.$$

$\square$

Let us now focus on the case of two antipodal singular points $p_1 = -p_2$. Given any point $p \in S^2 \subset \mathbb{R}^3$ we consider the space

$$H_{rad,p} := \left\{u \in H^1(S^2)\ :\ \exists\, \varphi : [-1,1] \longrightarrow \mathbb{R}\ \text{ measurable s.t. }\ u(x) = v(x \cdot p) \text{ for a.e. } x \in S^2\right\}.$$

Lemma 2.1. *Suppose $m = 2$, $\min\{\alpha_1, \alpha_2\} = \alpha_1 > 0$ and $p_2 = -p_1$. If h is a positive function satisfying* (2)*, then the Moser-Trudinger functional J^h_ρ is bounded from below on H_{rad,p_1} for any $\rho \in (0, 8\pi(1+\alpha_1))$.*

Proof. Let us consider

$$\widetilde{h}(x) := e^{-4\pi\alpha_1(G(x,p_1)+G(x,p_2))}.$$

Since $h = Ke^{-4\pi\alpha_1 G(x,p_1)-4\pi\alpha_2 G(x,p_2)} \leq \widetilde{h}\max_{x\in S^2} K(x)e^{4\pi(\alpha_1-\alpha_2)G(x,p_2)}$ it is sufficient to prove that the functional

$$\widetilde{J}_\rho(u) := J_\rho^{\widetilde{h}}(u) = \frac{1}{2}\int_{S^2}|\nabla u|^2 dv_{g_0} + \frac{\rho}{4\pi}\int_{S^2} u\, dv_{g_0} - \rho\log\left(\frac{1}{4\pi}\int_{S^2}\widetilde{h}e^u dv_{g_0}\right)$$

is bounded from below for any $\rho < 8\pi(1+\alpha_1)$. Let us consider Euclidean coordinates (x_1, x_2, x_3) on S^2 such that $p_1 = (0,0,-1)$, $p_2 = (0,0,1)$, and let π be the stereographic projection from the point p_2. Given a function $u \in H^1(S^2)$ we define $v(|y|) := (u(\pi^{-1}(y)))$, $v_{\alpha_1}(y) := v(|y|^{\frac{1}{1+\alpha_1}})$ and $u_{\alpha_1}(x) := v_{\alpha_1}(|\pi(x)|)$. Then we have

$$\int_{S^2}|\nabla u|^2 dv_{g_0} = 2\pi\int_0^\infty t|v'(t)|^2 dt = (1+\alpha_1)\int_0^{+\infty} s|v'_{\alpha_1}(s)|^2 ds = (1+\alpha_1)\int_{S^2}|\nabla u_{\alpha_1}|^2 dv_{g_0}, \quad (17)$$

and, using that $\sup_{t>0}\dfrac{1+t^{2(1+\alpha_1)}}{(1+t^2)^{1+\alpha_1}} < +\infty$,

$$\int_{S^2}\widetilde{h}e^u dv_{g_0} = 8\pi\int_0^{+\infty} e^{2\alpha_1}\frac{t^{2\alpha_1+1}e^{v(t)}}{(1+t^2)^{2(1+\alpha_1)}}dt \leq c_{\alpha_1}\int_0^{+\infty}\frac{t^{2\alpha_1+1}e^{v_{\alpha_1}(t^{1+\alpha_1})}}{(1+t^{2(1+\alpha_1)})^2}dt =$$

$$= 4\widetilde{c}_{\alpha_1}\int_0^{+\infty}\frac{se^{v_{\alpha_1}(s)}}{(1+s^2)^2} = \widetilde{c}_{\alpha_1}\int_{S^2} e^{v_{\alpha_1}}dv_{g_0}. \quad (18)$$

Finally, $\forall\, \varepsilon > 0$, $t \in \mathbb{R}^+$

$$|v(t) - v_{\alpha_1}(t)| \leq \left|\int_t^{t^{\frac{1}{1+\alpha_1}}}|v'(s)|ds\right| \leq \left|\int_t^{t^{\frac{1}{1+\alpha_1}}} s|v'(s)|^2 ds\right|^{\frac{1}{2}}\left|\frac{\alpha_1}{1+\alpha_1}\log t\right| \leq$$

$$\leq \frac{\varepsilon}{4\pi}\|\nabla u\|_2^2 + c_{\varepsilon,\alpha_1}|\log t|$$

from which

$$\left|\int_{S^2} u\, dv_{g_0} - \int_\Sigma u_{\alpha_1}\, dv_{g_0}\right| \leq 8\pi\int_0^{+\infty}\frac{|v(t)-v_{\alpha_1}(t)|}{(1+t^2)^2} \leq \varepsilon\|\nabla u\|_2^2 + C_{\varepsilon,\alpha_1}. \quad (19)$$

(17), (18), (19) and the Moser-Trudinger inequality (9) imply

$$\widetilde{J}_\rho(u) \geq (1+\alpha_1)\left(\frac{1}{2} - \rho\,\varepsilon\right)\int_{S^2}|\nabla u_{\alpha_1}|^2 dv_{g_0} + \rho\int_{S^2} u_{\alpha_1}dv_{g_0} - \rho\log\left(\frac{1}{4\pi}\int_{S^2} e^{u_{\alpha_1}}dv_{g_0}\right) - C_{\epsilon,\alpha_1,\rho} =$$

$$= (1+\alpha_1)\left(\left(\frac{1}{2} - \rho\,\varepsilon\right)\int_{S^2}|\nabla u_{\alpha_1}|^2 dv_{g_0} - \frac{\rho}{1+\alpha_1}\log\left(\frac{1}{4\pi}\int_{S^2} e^{u_{\alpha_1}-\overline{u}_{\alpha_1}}dv_{g_0}\right)\right) - C_{\epsilon,\alpha_1,\rho} \geq -\widetilde{C}_{\epsilon,\alpha_1,\rho}$$

if $\rho < 8\pi(1+\alpha_1)$ and ε is sufficiently small. $\square$

Remark 2.1. *Arguing as in [17], it is possible to describe the behavior of sequences of minimum points of J_ρ^h in $H^1_{rad,p_1}(S^2)$ as $\rho \nearrow 8\pi(1+\alpha_1)$ to prove that also $J^h_{8\pi(1+\alpha_1)}$ is bounded from below. Moreover if $K \equiv 1$ and $\alpha_1 = \alpha_2 = \alpha$ then we have*

$$\log\left(\frac{1}{4\pi}\int_{S^2} h e^{u-\overline{u}} dv_{g_0}\right) \le \frac{1}{16\pi(1+\alpha)}\int_{S^2} |\nabla u|^2 dv_{g_0} + \alpha - \log(1+\alpha) \quad \forall\, u \in H_{rad,p_1}(S^2),$$

with equality holding for

$$u \circ \pi^{-1}(y) = 2\log\left(\frac{(1+|y|^2)^{1+\alpha}}{1+e^\lambda |y|^{2(1+\alpha)}}\right) + c,$$

where $\lambda, c \in \mathbb{R}$ and π is the stereographic projection from p_1.

Proof of Theorem 1.2. By Lemma 2.1, $\forall\, \rho < 8\pi(1+\alpha_1)\ \exists\, \delta_\rho, C_\rho > 0$ such that

$$J^h_\rho(u) \ge \delta \int_{S^2} |\nabla u|^2 dv_{g_0} - C_\rho$$

$\forall\, u \in H_{rad,p_1}$. Thus J^h_ρ is coercive on the space

$$\left\{u \in H_{rad,p_1},\ \int_\Sigma u\, dv_{g_0} = 0\right\},$$

and by direct methods we can find a minimum point of J^h_ρ in $H^1_{rad,p}$. Since $h \in H^1_{rad,p_1}$, by Palais' criticality principle (see Remark 11.4 in [1]), this minimum point is a solution of (7). □

As a consequence of Theorems 1.1 and 1.2 we obtain a multiplicity result for equation (7). Indeed we can observe that if $\rho < 8\pi$ is sufficiently close to 8π, one has

$$\min_{u\in H^1(S^2)} J^h_\rho < \min_{u \in H_{rad,p_1}} J^h_\rho.$$

Corollary 2.1. *Suppose h satisfies the hypotheses of Theorem 1.2. There exists $\varepsilon_0 > 0$ such that $\forall\, \rho \in (8\pi - \varepsilon_0, 8\pi)$, equation (7) has at least two solutions u, v such that $u \in H_{rad,p_1}$ and $v \in H^1(S^2)\backslash H_{rad,p_1}$.*

Proof. For any $\rho < 8\pi$ let us take two functions $u_\rho \in H^1(S^2), v_\rho \in H_{rad,p_1}$, such that

$$J^h_\rho(u_\rho) = \min_{H^1(S^2)} J^h_\rho, \qquad J^h_\rho(v_\rho) = \min_{H_{rad,p_1}(S^2)} J^h_\rho(u) \quad \text{and} \quad \int_\Sigma u_\rho dv_{g_0} = \int_\Sigma v_\rho dv_{g_0} = 0.$$

We claim that, for ε sufficiently small and $\rho \in (8\pi - \varepsilon, 8\pi)$, $u_\rho \notin H_{rad,p_1}$ and in particular $u_\rho \ne v_\rho$. Assume by contradiction that there exists a sequence $\rho_n \nearrow 8\pi$ for which $u_{\rho_n} \in H_{rad,p_1}$. Then, applying Lemma 2.1 as in the proof Theorem 1.2, we would have

$$J^h_{\rho_m}(u_{\rho_m}) \ge \delta \int_{S^2} |\nabla u_{\rho_n}|^2 dv_{g_0} - C$$

for some $\delta, C > 0$. Therefore $\|\nabla u_{\rho_n}\|_2$ would be uniformly bounded and, up to subsequences, $u_{\rho_n} \rightharpoonup u$ in $H^1(S^2)$ with $J^h_{8\pi}(u) = \inf_{H^1(S^2)} J^h_{8\pi}$. This is not possible because we know by Theorem 1.1 that $J^h_{8\pi}$ has no minimum point. □

Now we will discuss some sufficient conditions for the existence of solutions of (7), without symmetry assumptions on h. Let H_0, T_ρ, d_ρ and $\Gamma(\alpha_1, \ldots, \alpha_m)$ be defined as in (10), (11), (12) and (13). First of all we recall a well known result concerning blow-up analysis for sequences of solutions.

Proposition 2.1 (See [2], [5])**.** *Let (Σ, g) be a compact Riemannian surface and let h be a function satisfying (8) with $K \in C^\infty_+(\Sigma)$. If u_n is a sequence of solutions of (7) on Σ with $\rho = \rho_n \longrightarrow \overline{\rho}$ and $\int_\Sigma u_n dv_g = 0$, Then, up to subsequences, one of the following holds:*

(i) $|u_n| \le C$ *with C depending only on $\alpha_1, \ldots, \alpha_m$, $\max_\Sigma K$, $\min_\Sigma K$ and $\overline{\rho}$.*

(ii) (blow-up). There exists a finite set $S = \{q_1, \ldots, q_k\}$ such that $u_n \longrightarrow -\infty$ uniformly on compact subsets of $\Sigma \backslash S$. Moreover $\frac{hu_n}{\int_\Sigma he^{u_n} dv_g} \rightharpoonup \sum_{i=1}^k \beta_i \delta_{q_i}$ with $\beta_i = 8\pi$ if $q_i \in \Sigma \backslash \{p_1, \ldots, p_m\}$ and $\beta_i = 8\pi(1 + \alpha_j)$ if $q_i = p_j$ for some $1 \le j \le m$.

Clearly case *(ii)* is possible only if $\overline{\rho} \in \Gamma(\alpha_1, \ldots, \alpha_m)$. As a direct consequence of Proposition 2.1 we get that, if E is a compact subset of $(0, +\infty) \backslash \Gamma(\alpha_1, \ldots, \alpha_n)$, the set of all the solutions of (7) in H_0 with $\rho \in E$ is a bounded subset H_0. This bound depends only on E, $\alpha_1, \ldots, \alpha_m$ and on $\max_\Sigma K, \min_\Sigma K$, thus, using the homotopy invariance of the Leray-Schauder degree, one can prove that, if R is chosen sufficiently large, d_ρ is well defined and does not depend on R and K. Moreover d_ρ is constant on every connected component of $(0, +\infty) \backslash \Gamma(\alpha_1, \ldots, \alpha_n)$. In [13] Chen and Lin introduced the generating function

$$g(x) := (1 + x + x^2 + x^3 \ldots)^{m-2} \prod_{i=1}^m (1 - x^{1+\alpha_i})$$

and observed that

$$g(x) = 1 + \sum_{j=1}^\infty b_j x^{n_j} \tag{20}$$

where $n_1 < n_2 < n_3 < \ldots$ are such that

$$\Gamma(\alpha_1, \ldots, \alpha_m) = \{8\pi n_j \; : \; j \ge 1\}.$$

Theorem D ([13])**.** *Let h be a function satisfying (8), then for $\rho \in (8\pi n_k, 8\pi n_{k+1})$ we have*

$$d_\rho = \sum_{j=0}^k b_j$$

where $b_0 = 1$ and b_j are the coefficients in (20).

As a consequence of this formula, (7) has a solution for any $\rho \in (0, 8\pi(1 + \alpha_1)) \backslash 8\pi\mathbb{N}$.

Lemma 2.2. *Suppose that h satisfies (8) with $K \in C^\infty_+(S^2)$, $m \ge 2$ and $0 < \alpha_1 \le \ldots \le \alpha_m$. Then equation (7) has a solution $\forall\, \rho \in (0, 8\pi(1 + \alpha_1)) \backslash 8\pi\mathbb{N}$.*

Proof. Indeed the first negative coefficient appearing in the expansion

$$g(x) = (1 + x + x^2 + x^3 \dots)^{m-2} \prod_{i=1}^{m} (1 - x^{1+\alpha_i}) = 1 + \sum_{j=1}^{\infty} b_j x^{n_j}$$

is the coefficient of $x^{1+\alpha_1}$, i.e.

$$g(x) = \sum_{j=0}^{\infty} b_j x^{n_j}$$

with $b_0 = 1$ and $b_j \geq 0$ for any $j \geq 1$ such that $n_j < 1 + \alpha_1$. From Theorem D it follows that $d_\rho \geq 1$ for $\rho \in (0, 8\pi(1+\alpha_1)) \backslash 8\pi\mathbb{N}$. $\square$

Remark 2.2. *Lemma 2.2 only holds for $m \geq 2$. Indeed for $m = 1$ and $K \equiv 1$ one can use a Pohozaev-type identity (see [17]) to prove that (7) has no solutions for $\rho \in [8\pi, 8\pi(1+\alpha_1)]$.*

Remark 2.3. *A different proof of Lemma 2.2 was given in [4] by Bartolucci and Malchiodi using topological methods.*

By Proposition 2.1, if $\rho_n \longrightarrow 8k\pi$ with $k < 1 + \alpha_1$, then any blowing-up sequence of solutions of (7) must concentrate around exactly k points $q_1, \dots, q_k \in \Sigma \backslash \{p_1, \dots, p_m\}$. A more precise description of the blow-up set is given in [10] (see also [12], [13]):

Proposition 2.2 ([10], [12])**.** *Let u_n be a sequence of solutions of (7) with $\rho = \rho_n \longrightarrow 8\pi k$ and $k < 1 + \alpha_1$. If alternative* (ii) *of Proposition 2.1 holds, then u_n has exactly k blow-up points $q_1, \dots, q_k \in \Sigma \backslash \{p_1, \dots, p_m\}$ and $(q_1, \dots, q_k)$ is a critical point of the function*

$$f_h(x_1, \dots, x_k) := \sum_{j=1}^{k} \left(\log h(x_j) + \sum_{l \neq j} G(x_l, x_j) \right)$$

in

$$\{(x_1, \dots, x_k) \in (S^2)^k \ : \ x_i \neq x_j \ \text{ for } i \neq j\}.$$

Moreover we have

$$\rho_n - 8k\pi = \sum_{j=1}^{k} h(q_{j,n})^{-1} \left(\Delta_{g_0} \log h(q_{j,n}) + 2(k-1) \right) \frac{\lambda_{j,n}}{e^{\lambda_{j,n}}} + O(e^{-\lambda_{j,n}})$$

where $q_{j,n}$ are the local maxima of u_n near q_j and $\lambda_{j,n} = u_n(q_{j,n})$.

Proof of Theorems 1.3 and 1.4. Take a sequence $\rho_n \searrow 8k\pi$ and a solution $u_n \in H_0$ of (7) for $\rho = \rho_n$. By Propositions 2.1, 2.2 and standard elliptic estimates, either u_n is uniformly bounded in $W^{2,q}(S^2)$ for any $q \geq 1$ or u_n blows-up at $(q_1, \dots, q_k) \in \Sigma \backslash \{p_1, \dots, p_m\}$. In the former case we have $u_n \longrightarrow u$ in $H^1(S^2)$ and u satisfies (7) with $\rho = 8\pi k$. The latter case can be excluded using (14), (15). Indeed we have

$$\Delta_{g_0} \log h(q_j) + 2(k-1) = \Delta_{g_0} \log K - \sum_{i=1}^{m} \alpha_i + 2(k-1) < 0$$

for any j. Denoting $q_{j,n}$ the maximum point of u_n near q_j and $\lambda_{j,n} = u_n(q_{j,n})$, by Proposition 2.2 we get

$$\rho_n - 8\pi k = \sum_{j=1}^{k} h(q_{j,n})^{-1} \left(\Delta_{g_0} \log h(q_{j,n}) + 2(k-1)\right) \frac{\lambda_{j,n}}{e^{\lambda_{j,n}}} + O(e^{-\lambda_{j,n}}) =$$

$$= \sum_{j=1}^{k} h(q_j)^{-1} \left(\Delta_{g_0} \log h(q_j) + 2(k-1)\right) \lambda_{j,n} e^{-\lambda_{j,n}} + o(\lambda_{j,n} e^{-\lambda_{j,n}}) < 0$$

which contradicts $\rho_n \searrow 8k\pi$. □

In order to prove Theorems 1.5, 1.6 we need to compute the Leray-Schauder degree for $\rho = 8\pi$.

Lemma 2.3. *Let h be a function satisfying* (8) *with $K \in C^\infty_+(\Sigma)$ and $\alpha_1, \ldots, \alpha_m > 0$. If $\Delta_{g_0} h(q) \neq 0$ for any $q \in \Sigma \backslash \{p_1, \ldots, p_m\}$ critical point of h, then $d_{8\pi}$ is well defined.*

Proof. It is sufficient to prove that the set of solutions of (7) in H_0 with $\rho = 8\pi$ is a bounded subset of H_0. Assume by contradiction that there exists $u_n \in H_0$ solution of (7) for $\rho = 8\pi$ such that $\|u_n\|_{H_0} \longrightarrow +\infty$. By Propositions 2.1 and 2.2, there exists $q \in \Sigma \backslash \{p_1, \ldots, p_m\}$ such that $u_n \rightharpoonup 8\pi\delta_q$, $\nabla h(q) = 0$ and

$$0 = h(q_n)^{-1} \Delta_{g_0} \log h(q_n) \lambda_n e^{-\lambda_n} + O(e^{-\lambda_n}) = h(q)^{-2} \Delta_{g_0} h(q) \lambda_n e^{-\lambda_n} + o(\lambda_n e^{-\lambda_n})$$

where $\lambda_n := \max_\Sigma u_n$ and $u_n(q_n) = \lambda_n$. Since $\Delta_{g_0} h(q) \neq 0$ this is not possible. □

Under nondegeneracy assumptions, Chen and Lin proved that for any critical q point of h there exists a blowing-up sequence of solutions which concentrates at q. Moreover they were able to compute the total contribution to the Leray-Schauder degree of all the solutions concentrating at q.

Proposition 2.3 (see [11], [13])**.** *Assume that h is a Morse function on $\Sigma \backslash \{p_1, \ldots, p_m\}$. Given a critical point $q \in \Sigma \backslash \{p_1, \ldots, p_m\}$ of h, the total contribution to $d_{8\pi}$ of all the solutions of* (7) *concentrating at q is equal to* $\mathrm{sgn}(\rho - 8\pi)(-1)^{\mathrm{ind}_p}$*, where* ind_p *is the Morse index of p as critical point of h.*

Proof of Theorems 1.5, 1.6. Let us denote

$$\Lambda_- = \{q \in \Sigma \backslash \{p_1, \ldots, p_m\} \ : \ \nabla h(q) = 0, \ \Delta_{g_0} h(q) < 0\},$$

$$\Lambda_+ = \{q \in \Sigma \backslash \{p_1, \ldots, p_m\} \ : \ \nabla h(q) = 0, \ \Delta_{g_0} h(q) > 0\}.$$

By Proposition 2.3 we have

$$d_{8\pi} = 1 - \sum_{q \in \Lambda_-} (-1)^{\mathrm{ind}_q} = \overline{d} + \sum_{q \in \Lambda_+} (-1)^{\mathrm{ind}_q},$$

where $\overline{d}$ is the Leray-Schauder degree for $\rho \in (8\pi, 8\pi + \varepsilon)$. Clearly Λ_- contains only the local maxima of h and the saddle points of h in which $\Delta_{g_0} h < 0$, thus

$$d_{8\pi} = 1 - r + s.$$

Therefore we get existence of solutions if $r \neq s + 1$. Similarly we have

$$d_{8\pi} = \overline{d} - s' + r'$$

and we get solutions if $s' \neq r' + \overline{d}$. $\overline{d}$ can be computed using Theorem D. If $m \geq 2$,

$$g(x) = 1 + x + \cdot \ldots \quad \Longrightarrow \quad \overline{d} = 2.$$

If $m = 1$ we have

$$g(x) := 1 - x - x^{1+\alpha} + x^{2(1+\alpha)} \quad \Longrightarrow \quad \overline{d} = 0.$$

If $m = 0$, then

$$g(x) = 1 - 2x + x^2 \quad \Longrightarrow \quad \overline{d} = -1.$$

This concludes the proof. □

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